Noise in stochastic cellular automata

Doi-Peliti approach

Consider a lattice cellular automata defined by some local stochastic CA rule $\mathcal{A}$. Let $m_{\mathbf{r}}$ be the operator that measures the value of the spin at site $\mathbf{r}$ (with eigenvalues in $\{\pm 1\}$), and let $n_{\mathbf{r}} = \frac{1+m_{\mathbf{r}}}{2}$. Define the raising ($a_{\mathbf{r}}^\dagger$) and lowering ($a_{\mathbf{r}}$) operators as usual to satisfy

$$[n_{\mathbf{r}}, a_{\mathbf{r}'}] = -\delta_{\mathbf{r},\mathbf{r}'} a_{\mathbf{r}}, \qquad [n_{\mathbf{r}}, a_{\mathbf{r}'}^\dagger] = \delta_{\mathbf{r},\mathbf{r}'} a_{\mathbf{r}}^\dagger. \tag{1}$$

Finally, let $\langle +|$ be the uniform sum over all spin configurations. Note that $\langle +|a_{\mathbf{r}}^\dagger = \langle +|(1-n_{\mathbf{r}})$ and $\langle +|a_{\mathbf{r}} = \langle +|n_{\mathbf{r}}$.

Suppose $\mathcal{A}$ has l different types of transitions, which it applies with probabilties q_a, $a = 1, \dots, l$. Let $\Pi_{\mathbf{r}}^{\mp\to\pm,a}$ be the projector onto states where the ath rule would send a $\pm$ spin to a $\mp$ spin. We can write the Hamiltonian generating the dynamics (via. $\partial_t|P\rangle = -H|P\rangle$) as

$$H = -\sum_{\mathbf{r}} \sum_{a=1}^{l} q_a \left((a_{\mathbf{r}}^\dagger - (1-n_{\mathbf{r}}))\Pi_{\mathbf{r}}^{a,-\to+} + (a_{\mathbf{r}} - n_{\mathbf{r}})\Pi_{\mathbf{r}}^{a,+\to-} \right). \tag{2}$$

Taking the expectation value of a single commutator of an operator $\mathcal{O}$ with H gives the noise-averaged time derivative of $\mathcal{O}$, and taking a double commutator of $\mathcal{O}$ with H gives the variance of the noise appearing in the Langevin equation for $\mathcal{O}$. To compute the dynamics of the magnetization, we thus just need the relations

$$\begin{aligned} \langle +|[n_{\mathbf{r}}, H] &= -\langle +|\sum_a q_a(\Pi_{\mathbf{r}}^{a,-\to+} - \Pi_{\mathbf{r}}^{a,+\to-}) \\ \langle +|[n_{\mathbf{r}'}, [n_{\mathbf{r}}, H]] &= -\delta_{\mathbf{r},\mathbf{r}'}\langle +|\sum_a q_a(\Pi_{\mathbf{r}}^{a,-\to+} + \Pi_{\mathbf{r}}^{a,+\to-}) \end{aligned} \tag{3}$$

To derive these, we used some of the above relations and the fact that e.g. $n_{\mathbf{r}}\Pi_{\mathbf{r}}^{a,+\to-} = \Pi_{\mathbf{r}}^{a,+\to-}$. Taking the inner product of the above expressions with the steady-state probability distribution gives $\langle p_\theta n_{\mathbf{r}}\rangle$ and $\langle \partial_t n_{\mathbf{r}} \partial_{t'} n_{\mathbf{r}'}\rangle_c$, respectively. This means that the Langevin equation for $n_{\mathbf{r}}$ is

$$\langle \partial_t n_{\mathbf{r}} \rangle = -\sum_a q_a \langle \Pi_{\mathbf{r}}^{a,-\to+} - \Pi_{\mathbf{r}}^{a,+\to-}\rangle + \xi_{\mathbf{r}}(t), \tag{4}$$

where the noise $\xi_{\mathbf{r}}(t)$ has correlations

$$\langle \xi_{\mathbf{r}}(t)\xi_{\mathbf{r}'}(t')\rangle = \delta_{\mathbf{r},\mathbf{r}'}\delta_{t,t'} \sum_a q_a \langle \Pi_{\mathbf{r}}^{a,-\to+} + \Pi_{\mathbf{r}}^{a,+\to-}\rangle. \tag{5}$$

Generating function approach

Define the generating function

$$W[z] = \ln \left\langle \prod_{\mathbf{r},t} e^{z_{\mathbf{r}}(t)\delta n_{\mathbf{r}}(t)} \right\rangle, \tag{6}$$

where $\delta n_{\mathbf{r}}(t) \in \{-1, 0, +1\}$ is the jump in the value of $n_{\mathbf{r}}$ at time step t. Connected correlation functions of $\partial_t n_{\mathbf{r}}$ can be computed by taking functional derivatives of W and then setting $z = 0$. Suppose that the CA dynamics occurs at a given site and time step with probability $dt \to 0$. Then we may write

$$W[z] = \ln \left\langle \prod_{\mathbf{r},t} \left(1 + dt \sum_a q_a \left((e^{z_{\mathbf{r}}} - 1)\Pi_{\mathbf{r}}^{a,-\to+} + (e^{-z_{\mathbf{r}}} - 1)\Pi_{\mathbf{r}}^{a,+\to-} \right) \right) \right\rangle. \tag{7}$$

Since the applications of the CA dynamics at distinct space time points are independent events (each site updating according to independent Poisson clocks), we can take the expectation value inside the product. In the $dt \to 0$ limit, we can then write

$$W[z] = \sum_{\mathbf{r}} \int_t \sum_a q_a \left\langle (e^{z_{\mathbf{r}}} - 1)\Pi_{\mathbf{r}}^{a,-\to+} + (e^{-z_{\mathbf{r}}} - 1)\Pi_{\mathbf{r}}^{a,+\to-} \right\rangle. \tag{8}$$

Taking first and second functional derivatives of W wrt z and setting $z = 0$ is then easily checked to give the same Langevin equation as before.

Examples

Sanity check: just noise

Consider the trivial case where a single spin is subject to noise of strength p and bias η. Then

$$\Pi^{\mp\to\pm} = \frac{1 \mp m}{2}, \tag{9}$$

with these transitions appearing with probability $q_{\mp\to\pm} = p\frac{1\pm\eta}{2}$. Thus, after tracking down some factors of 2, we get

$$\partial_t m = p(\eta - m) + \xi, \tag{10}$$

where

$$\langle \xi(t)\xi(t') \rangle = 2p(1 - \eta m). \tag{11}$$

The correlations of the noise make sense: if $\eta = \pm 1$, then the state with $m = \pm 1$ is absorbing, and not subject to noise.

R *squeezing code*

Now consider the R squeezing code. Let $\bar{n}_{\mathbf{r}} = 1 - n_{\mathbf{r}}$. Let $a = 1$ be the projectors associated with the noiseless dynamics, and $a = 2$ the projectors associated with the noise (which are the same as in the trivial example above). The former are

$$\begin{aligned} \Pi_{\mathbf{r}}^{1,-\to+} &= (1 - \bar{n}_{\mathbf{r}+\hat{\mathbf{x}}}\bar{n}_{\mathbf{r}-\hat{\mathbf{x}}})\bar{n}_{\mathbf{r}} \\ \Pi_{\mathbf{r}}^{1,+\to-} &= (1 - n_{\mathbf{r}+\hat{\mathbf{y}}}n_{\mathbf{r}-\hat{\mathbf{y}}})n_{\mathbf{r}}, \end{aligned} \tag{12}$$

each appearing with strength $(1-p)/2$. The variance of the noise is therefore, after a bit of algebra,

$$\langle \xi_{\mathbf{r}}(t)\xi_{\mathbf{r}'}(t')\rangle = \delta_{t,t'}\delta_{\mathbf{r},\mathbf{r}'}\Big\langle 2p(1-\eta m_{\mathbf{r}}) + \frac{1-p}{4}\Big((1-m_{\mathbf{r}})(3+m_{\mathbf{r}+\hat{\mathbf{x}}}+m_{\mathbf{r}-\hat{\mathbf{x}}}-m_{\mathbf{r}+\hat{\mathbf{x}}}m_{\mathbf{r}-\hat{\mathbf{x}}}) \\ + (1+m_{\mathbf{r}})(3-m_{\mathbf{r}+\hat{\mathbf{y}}}-m_{\mathbf{r}-\hat{\mathbf{y}}}-m_{\mathbf{r}+\hat{\mathbf{y}}}m_{\mathbf{r}-\hat{\mathbf{y}}})\Big)\Big\rangle. \tag{13}$$

What to make of this rather complicated expression? I'm not really sure. Note that even when $p=0$, the ideal CA updates produce noise for the magnetization by virtue of their stochasticity (unless $m=\pm 1$, for which cases the noise variance correctly vanishes when $p=0$). It is also worth noting that if we were to take a mean-field factorization, we would get

$$\sigma_{\mathbf{r}}^2 = \frac{3-p}{2} + O(\langle m\rangle), \tag{14}$$

meaning that when $\langle m\rangle$ is small, the noise actually *de*creases as p is *in*creased.