Linear exponential suppression of the logical error rate under union-find decoding

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For the union-find (UF) decoder on the $L \times L$ square torus, we show that the logical error probability is $p^{\Theta(L)}$ at sufficiently small edge-error probability p, with perfect syndrome measurements (extensions to faulty measurements and full detector graphs are straightforward).

This result improves in some respects the proof of a threshold for UF in [3], which provided a bound on the logical error probability of $p^{\Omega(L^{\eta/\log\log d})}$ with constant η. The proof in [3] was based on Gacs' clustering methods, and, as far as I can tell, this prevents it from achieving the full $p^{\Theta(L)}$ scaling that we all believed to be the correct answer. When [3] was written, I had thought for several days about using a Peierls argument to prove $p^{\Theta(L)}$, but the sketch of my final argument did not hold up under scrutiny by my coauthors. Naturally I was later compelled to turn AI models on this question, and Astra 6.2 provided with my notes produced a suitable proof, which appears below.

1 Statement and decoder

Work in one Pauli sector of the $L \times L$ square toric code, $L \geq 3$. Let E be the set of erroneous edges. The measured defects are the vertices incident to an odd number of edges of E and form its boundary ∂E. A correction is an edge set with that same boundary. Its symmetric difference with E is a cycle. Decoding fails if this cycle winds an odd number of times around the torus in at least one periodic direction. Let H be the lattice with a midpoint inserted in each edge. Distances and tree lengths below count edges of H (half-edges of the original lattice). Let F contain both halves of each edge of E. Then $\partial F = \partial E$, and midpoints always have even fault degree.

In synchronous half-edge UF, the parity of a component is the number of defects it contains modulo two. Initially each defect is an odd singleton. In a round, each odd component adds its neighbors in H, even components stay still, and overlapping sets merge simultaneously. At the end, choose any syndrome-compatible correction supported in the grown components. For example, choose a tree spanning each final component. Remove its leaves one at a time. If the removed leaf carries a defect, include its edge to the parent in the correction and toggle the defect at the parent. Even parity at termination ensures that the root clears. Equivalently, keep a partition of *all* vertices of H starting with singletons. In each round join old parts along every edge between different parts with at least one odd endpoint part. Take connected components of these joins. An untouched nondefect singleton simply stands for unclaimed space. Indeed, a

newly reached vertex is either in an old part or is such a singleton, and two growing parts reaching the same new vertex merge through it. This is the incidence-graph formulation of synchronous half-edge growth used in [3].

Theorem 1. *There are absolute constants $a, A, p_* > 0$ such that, for independent Bernoulli-p edge errors and $0 < p \le p_*/4$,*

$$p^L \;\le\; p_{\mathsf{log}}(L,p) \;\le\; AL^2(p/p_*)^{aL}. \tag{1}$$

Here failure means that the error plus the correction has nonzero homology over $\mathbb{F}_2$. The upper bound also holds when $\Pr(S \subseteq E) \le p^{|S|}$ for every original edge set S.

Delfosse and Nickerson [1] proved that UF corrects every error of weight at most $\lfloor (L-1)/2 \rfloor$. Yoshida, Lake and Yamasaki [3, Theorem 3] proved a finite threshold for the rotated surface code under circuit-level local stochastic noise, with a suppression exponent $\Omega(d^{\eta/\log\log d})$ in the code distance d. Theorem 1 gives a linear exponent for the two-dimensional periodic graph with perfect measurements. The counting step follows the Peierls approach used for topological quantum memories by Dennis, Kitaev, Landahl and Preskill [2].

How a logical failure leaves evidence in the noise

Consider a winding loop in the error plus the correction. Follow it across the lattice. Between final UF clusters, every step must be a physical fault, since the decoder only corrects inside its clusters. Inside a cluster the correction may cross a long distance. A long run of faults, one large cluster, or a succession of shorter runs and clusters can each carry the loop across the torus. We need to account for the total distance in all three cases.

Why should distance across a cluster cost faults? An odd cluster expands each round. When it absorbs a previously even cluster it can resume growth from the far side, skipping an entire region in one round. Trace a defect whose cluster stayed odd, and temporarily shrink each even cluster it absorbed to a point. In this graph the defect is the only vertex of odd fault degree inside its cluster. A path of faults must therefore lead from the defect to the outside. That same path crosses the cluster boundary as it stood before each round of growth. Each crossing edge becomes internal as soon as the cluster grows across it, so different rounds require distinct faults. Figure 2 shows how one such path can use pieces of two different error strings.

It remains to account for the skipped regions and locate the faults just identified. For each even cluster crossed, trace a longest path in the spanning tree built as its parts merged and apply the same argument to the older clusters along it. A cluster can supply a shortcut both on this tree path and on a path used to locate faults. Describe its history once. For each crossing, record just the displacement between the entry and exit vertices. A displacement across a cluster of diameter D takes $O(\log(2+D))$ bits in two dimensions. Older clusters appear strictly earlier in the growth history, which lets us bound the sum of these costs even when the history is deeply nested.

The outcome is a set of w marked half-edges whose locations, given a starting vertex, can be specified in $O(w)$ bits. A winding loop requires $w = \Omega(L)$. There are thus at most K^w choices per starting vertex for some constant K, and each choice

requires at least $w/2$ original edge faults. At small p, the factor $p^{w/2}$ dominates this count. This argument allows the clusters and the connecting error strings to depend on the same noise realization.

2 A merger history with one surviving seed

We build an auxiliary spanning tree for each part, preserving old trees at every merger. Its role is to track geometry. An odd part carries one *live seed*, an original defect. A nonsingleton even set stored by the construction will be called a *block*. We also allow ordinary passive singletons, with diameter zero. Every block has a birth round and a tree. Blocks may be genuine even UF parts or even subsets recorded at a simultaneous merger. The latter are used only in the proof.

In round h, choose a spanning tree of the contact graph of each merging group of old parts. For every tree edge keep a physical edge of H witnessing the contact. There are no passive–passive contact edges. Write m for the number of old odd parts in this tree.

- For even m the whole tree is one new block. Its m seeds retire.
- For $m = 1$ the unique seed survives. All the passive neighbors attach directly to its old part.
- For odd $m \geq 3$, delete the passive leaves temporarily. The remaining tree contains all m odd parts and has an odd leaf U, since each remaining passive vertex has degree at least two. Keep the seed of U. In the original tree remove U together with its pendant passive neighbors. The remainder is connected, has $m - 1$ odd parts, and forms one new even block. Its seeds retire. The new block and the removed passive neighbors each attach to the old U by one tree edge.

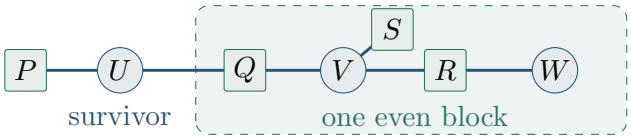


Figure 1: A simultaneous merger of three odd parts (circles). The squares are old passive parts. After trimming the passive leaves P, S, one may choose U as the surviving odd leaf. Restoring the leaves attaches P directly to U. All the parts in the dashed box attach through the single edge UQ. Their two seeds retire at this round.

Figure 1 shows the last case. An odd part without contacts retains its seed. Unmerged passive parts persist. In particular a live seed has belonged to an actual odd part at the start of *every* round since initialization. Distinct retirements involve distinct seeds. Every nonsingleton block is formed from a tree of old parts having at least two retiring seeds. A block used as an old passive part, or already absorbed by an old odd part, was born in a strictly earlier round.

Let G be an old odd part whose seed s retires in round h. Its vertex set consists of s and the disjoint passive sets directly attached to its continuing seed at earlier rounds.

Call the nonsingleton sets in this list its *private blocks*. In the tree of G, contract each private block to a vertex. Any vertex of the resulting tree is at distance at most $h-1$ from s. Indeed, each round attaches new sets directly to the pre-round part. Each such set, including an ordinary singleton, becomes a single vertex of the contracted tree. Its attachment adds at most one to the previous maximum depth. This applies also to the even remainder in the odd multiple-merger rule.

3 Paying for the lifetime of a seed

Lemma 1. For every retiring seed s in its pre-round part G at round h, there is a path P_s of marked edges (edges of F) in the graph with the private blocks of G contracted. It starts at s, stays in G up to its last edge, and ends outside G. It is simple in the quotient and $|P_s| \geq h$. Each edge on the path denotes a specific physical half-edge.

Proof. Contract the private blocks in the fault multigraph, identifying all their vertices even when the internal fault graph is disconnected. Delete loops, keeping physical representatives of parallel edges. Each contracted vertex has even fault degree by the parity of the block. Every uncontracted vertex of G except s is a nondefect singleton, so s is the only odd vertex inside G in the quotient. Its fault component must reach outside G by the handshake lemma. Take a simple path and truncate it at its first exit.

Let U_j be the actual part carrying s at the start of round j, for $1 \leq j \leq h$. It contains s, lies in G, and is a union of whole classes of the contraction. Each private block either has already been attached or is entirely outside U_j. Thus P_s crosses δU_j on a physical marked edge. Every edge of this cut has both endpoints in the same actual UF part by the end of round j, because U_j is odd and grows. It can never cross a later part boundary. The h cuts therefore use distinct edges of this one path. □

Figure 2 gives an example where the path switches between two original error strings inside the contraction.

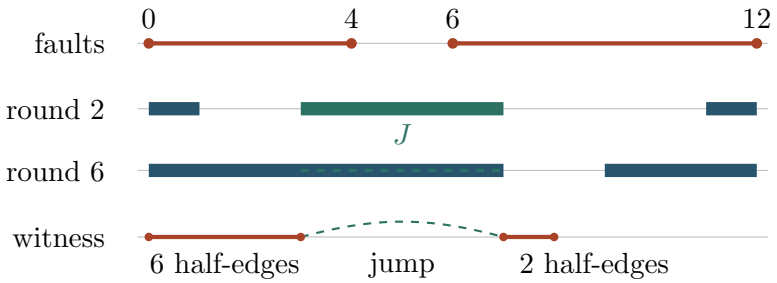


Figure 2: A row of a large torus, cropped to $0 \leq x \leq 12$. The only errors are the two red strings $[0, 4]$ and $[6, 12]$. At round 2 the inner defects form an even block J, whose intersection with the row is $[3, 7]$. At round 6 the cluster of $s = 0$ absorbs J. Its next contact with an odd cluster is at $x = 8$ in round 8. Contracting J gives a path from s to its first exit which uses six half-edges of the first fault string and two of the second. The two strings are physically disjoint.

We shall select only some retiring seeds. For *any* such selection, the paths with their last exit edges removed are pairwise edge-disjoint as physical edge lists. Indeed,

two pre-retirement parts are disjoint or nested, since actual parts only merge. In the nested case the earlier seed's entire old part entered an even block when it retired. That block, or a containing block, is inside one private block of the later seed if the later seed's part contains the earlier part. Its internal edges disappear in the later contraction. At a common retirement round the pre-round parts are disjoint. Finally, the first edge of each P_s is incident to its original seed. Distinct seeds give distinct first half-edges, because each half-edge has only one original vertex as endpoint. Thus if w distinct physical half-edges are marked, including all these paths,

$$\#\{s\} \le w, \qquad \sum_s h_s \le \sum_s |P_s| \le w + \#\{s\} \le 2w. \tag{2}$$

4 Certifying an even block

Let D_B be the intrinsic diameter of the auxiliary tree of block B. Select two endpoints of a tree diameter and follow their unique tree path. The following procedure selects child blocks, seeds, and walks. It will be used whenever a higher-level walk needs to cross B between specified ports.

Project the diameter path onto the tree of old parts that formed B. The projection is a simple path, possibly a point. If it visits no old odd part, it lies in one old passive block J. Then $D_B = D_J$, because the old tree is a subtree and contains a diameter of B. Replace B by J and repeat. A nonsingleton block of positive diameter cannot reduce to a singleton this way. Write B' for the first retained block. All ports in the original B are within tree distance $D_B = D_{B'}$ of every vertex of B', measured in the original B. Thus this reduction requires only one coordinate displacement from an original port to a diameter endpoint of B', however many equal-diameter blocks were skipped.

Now suppose the projected path visits $g \ge 1$ old odd parts. Select the retiring seed in each, all with lifetime h, the birth round of B. The projected path has at most $2g$ joining edges, since each edge is incident to an odd part and the path visits each part once. In each visited odd part G, the portion of the original diameter path uses at most $2(h-1)$ edges after contraction of its private blocks, by the depth bound. An old passive block directly on the projected path, and any private block met by the original tree path, is traversed at most once on that path. This follows from connectedness of its subtree. Hence

$$D_B \le 2gh + \sum_{J \in \mathcal{J}_B} D_J \tag{3}$$

whenever $\mathcal{J}_B$ includes all these nonsingleton blocks, each listed once. Extra children in $\mathcal{J}_B$ only increase the right-hand side.

To locate the marked edges we take a slightly longer walk. Follow the projected path. Inside each visited G, go from its entry port to s along the tree, take the fault path P_s out and back, then go from s to the exit port along the tree. Use the tree with private blocks contracted on each tree leg. A passage through a contracted block is a jump between the actual entry and exit vertices in that block. Record the actual half-edges of P_s as marked on the outward leg. For an old passive block directly on the

projected path, jump between its two ports. Endpoints lying in a block count as ports, including when the two ports coincide.

Take $\mathcal{J}_B$ to be the distinct nonsingleton blocks encountered on these walks in the specified contractions. The two seed tree legs together visit every private block needed by the original diameter path, since the unique tree path between two ports lies in the union of the two paths to the seed. Thus (3) applies. Each private block is owned by only one visited G. It is visited at most once on each of the two simple tree legs and once on each of the two fault legs. The last fault edge is immediately retraced and needs no traversal of the part in which it ends. Each directly visited passive block needs only one jump. Consequently each selected child has at most four passages at this parent. Recurse *once* for each distinct child, allowing the equal-diameter reduction at the start of the procedure. The recursion is finite because child birth rounds strictly decrease.

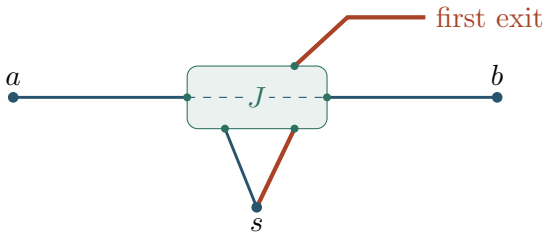


Figure 3: One visited odd part, with entry a, exit b, and private block J. The a–b tree path incurs D_J at most once. To locate the faults, the record goes a to s along the tree (blue), out and back on P_s (red), and s to b along the tree. This can give four different passages through J. The record describes J once and specifies the displacement for each passage.

Figure 3 illustrates the two walks. Equation (3) charges a child's diameter once, using the original diameter path. The longer walk locates the faults and may jump through that child several times. Only the logarithmic cost of a displacement is incurred on repeated jumps.

5 From a logical cycle to a count of certificates

A nontrivial cycle in the error plus the correction has a connected component with two vertices at ambient distance at least $\lfloor L/2 \rfloor$ in H. Indeed a winding cycle projects onto a full periodic coordinate. These vertices are connected in the union of F and the auxiliary trees of the final parts. Replacing a correction segment in a part by a path in its tree preserves the connectivity needed here. Contract the final trees and take a simple path between the two vertices' classes. The path may have zero edges. Its k edges are marked physical half-edges between different final parts. Each visited final nonsingleton block occurs once. Start the recursive procedure on those blocks, using the actual ports of this top path. If $\mathcal{J}_0$ denotes these blocks, then

$$\lfloor L/2 \rfloor \le k + \sum_{J \in \mathcal{J}_0} D_J. \tag{4}$$

The right side bounds the length of a connecting walk obtained by lifting the top path through the trees.

After skipping the equal-diameter steps, the selected blocks form a forest. Siblings are disjoint, and recursion never issues two calls for the same child. A selected seed retires in exactly one of the retained blocks. Let w count distinct marked half-edges, including those on the top path, and put

$$b_B = \sum_{s \text{ selected at } B} (1 + h_s + |P_s|), \qquad M_B = \sum_{J \text{ in the subtree at } B} b_J, \qquad M = 1 + k + \sum_B b_B. \tag{5}$$

We have $k \le w$. By (2), $M \le 1 + 6w$. Induction in (3) gives $D_B \le 2M_B$, including the diameters of its skipped representatives, and (4) gives $\lfloor L/2 \rfloor \le 2M$. In particular $w \ge cL$ for all sufficiently large L and an absolute $c > 0$.

Lemma 2. For a given starting vertex, the marked sets arising by this construction with $w \ge 1$ have at most K^w possibilities, for an absolute K.

Proof. We give a binary description from which all the marked half-edges can be read. The reader need only recover the marked set. Work in doubled square coordinates modulo $2L$. An ordinary lattice step uses a constant-size direction and a mark bit. A jump between two ports in a block of tree diameter D uses $O(1 + \log_2(1 + D))$ bits. For example, encode each signed coordinate difference with a sign, the binary digit count written in unary, and those digits. Choose a shortest coordinate representative modulo $2L$.

Record the top walk, jumping over final blocks. At the first port of each distinct selected block insert a subrecord. In that subrecord specify the displacement to the chosen diameter endpoint of the retained representative, then write its longer walk as in the previous section, inserting its child subrecords in the same way. An end symbol returns the reader to the saved port. The caller can then write its port-to-port jump. At later visits only the jump is written. Fixed finite tags distinguish steps, jumps, subrecord openings, returns, and termination. Each command either supplies its coordinates or recovers a position already on the recursion stack. Child identifiers and lists matching ports to blocks can therefore be omitted. A zero-length passage may be written as a zero displacement. Ordinary singleton parts need no jump or subrecord.

There are $O(M)$ ordinary steps and fixed overhead symbols besides those for block passages. Indeed at a retained block they come from at most $2g$ joining edges, at most $2g(h-1)$ tree steps, and twice the total lengths of its selected fault paths. Each retained block has positive b_B. Each is charged for at most four caller passages and one displacement to its own diameter endpoint. Skipping a chain incurs a single initial displacement bounded by its common diameter, and the caller's port-to-port displacements obey that same bound. The total bit length is therefore at most a constant times

$$M + \sum_B \log_2(1 + M_B). \tag{6}$$

For completeness, this last sum is $O(M)$ even for highly unbalanced histories. If h_B is the birth round of a retained block, $h_B \le b_B$ since at least one seed of that lifetime

is selected. Blocks of a fixed round h form an antichain, so their subtree masses sum to at most M. Using $\log_2(1+x) \le h + x2^{-h}/\ln 2$ for $h \ge 1$, we get

$$\sum_B \log_2(1+M_B) \le \sum_B h_B + \frac{1}{\ln 2}\sum_{h\ge 1} 2^{-h} \sum_{B:h_B=h} M_B \le (1+1/\ln 2)M. \tag{7}$$

Thus the bit length is $O(w)$, with an absolute constant. Every valid description, read from the specified starting vertex, identifies a fixed set of physical marked edges. Counting all binary words up to this length proves the claim. □

Proof of Theorem 1. There are $3L^2$ possible starting vertices in H. A fixed set of w marked half-edges requires at least $w/2$ distinct original faults. Its probability is at most $p^{w/2}$. The union bound and Lemma 2 give, for sufficiently large L,

$$p_{\mathsf{log}} \le 3L^2 \sum_{w\ge cL} (K\sqrt{p})^w. \tag{8}$$

Choosing $p_* = K^{-2}$ and summing proves the stated upper bound with $a = c/2$. For the finitely many smaller L at least one fault is necessary, so decreasing a and increasing A if necessary makes the same statement valid for all $L \ge 3$.

For the lower bound fix a noncontractible row of L edges and condition on all faults off the row. An empty row and a fully faulty row give the same syndrome, and the resulting error chains differ by a logical operator. For any fixed decoder randomness at least one of them fails. For $p \le 1/2$ the smaller of their conditional probabilities is p^L. Averaging proves the lower bound. The upper and lower bounds give $p_{\mathsf{log}} = p^{\Theta(L)}$ uniformly at sufficiently small p, after absorbing the polynomial factor. □

6 The cluster-radius estimate

How likely is a particular site to belong to a large final cluster? Fixing the site before drawing the noise removes the sum over starting vertices.

Corollary 1. *Fix a vertex x of H, and let $B(x)$ be its final UF part, with $D_{B(x)} = 0$ for a singleton. There are absolute constants $C, c_0 > 0$ such that, uniformly in x and L, for $r \ge 1$ and $0 < p \le p_*/4$,*

$$\Pr\big(D_{B(x)} \ge r\big) \le C(p/p_*)^{c_0 r}. \tag{9}$$

Proof. Start the record for $B(x)$ at x. The displacement to the first retained diameter endpoint costs $O(1+\log(1+D_{B(x)}))$ bits, already included in Lemma 2. There are no external connecting edges. Thus $D_{B(x)} \le 2\sum_B b_B \le 10w$ by (2). Counting the words from this single anchor gives $\sum_{w\ge r/10}(K\sqrt{p})^w$, with K enlarged throughout if needed. □

For example, if the cluster through x reaches a site at lattice distance r from x (measured in half-edges), its tree contains a path of length at least r. The corollary therefore bounds the probability of this event by $C(p/p_*)^{c_0 r}$. The bound gives exponential decay at small p. Determining the optimal coefficient in the exponent remains open.

References

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