Some dynamic RG calculations

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These notes contain some calculations related to doing dynamic RG within the MSR formalism. Thanks to Ruihua Fan for helping to check things and catching some typos.

Consider an order parameter field ϕ which obeys some Langevin equation

$$\partial_t \phi = L[\phi] + \eta, \tag{1}$$

where the noise η has correlation functions $\langle \eta\eta \rangle = 2D\delta\delta$. In this note we will develop some familiarity with how to use the MSR formalism to run RG on the couplings appearing in $L[\phi]$. The starting point for this formalism is a standard writing of the probability distribution $P(\phi)$ as

$$P(\phi) \propto \int \mathcal{D}i\psi \, e^{-\int \mathcal{L}[\phi,\psi]}, \tag{2}$$

where the integral is over spacetime, the i in $\mathcal{D}i\psi$ pendantically indicates that the response field ψ is to be integrated along the imaginary axis, and the Lagrangian (or more correctly, the action density) is[1]

$$\mathcal{L} = \psi(\partial_t \phi - L[\phi]) - D\psi^2. \tag{3}$$

The advantage of this approach is that to do RG, we can simply split our fields (the order parameter ϕ and the response field ψ) into fast $\phi_>, \psi_>$ and slow $\phi_<, \psi_<$ parts and integrate out the former, without expending much brainpower thinking about the meaning of what we are doing. Specifically, we need only calculate

$$\mathcal{L}_{\rm eff} = \mathcal{L}_< + \langle \mathcal{L}_{<>} \rangle_> - \frac{1}{2} \langle \mathcal{L}_{<>}^2 \rangle_{>,c} + \cdots \tag{4}$$

where $\mathcal{L}_{<>}$ is the part of the MSR Lagrangian that mixes the slow and fast modes, and $\langle \cdot \rangle_>$ denotes an average over the fast modes (with $\langle \cdot \rangle_{>,c}$ naturally being the connected part).

warmup I: Wilson-Fisher

We first consider an equilibrium example, with Langevin equation[2]

$$\partial_t \phi = \nu \nabla^2 \phi - r\phi - \frac{g}{3!}\phi^3 + \eta. \tag{5}$$

Since we may write the Langevin equation as $\partial_t \phi = -\delta F/\delta\phi + \eta$ for a well-defined F, we are dealing with dynamics that relaxes to equilibrium in a way that respects detailed balance, and the dynamic RG formalism will only add on to—rather than modify—the results in the equilibrium case. The point of doing this example is therefore just to make sure that we know how to turn the crank, and to determine the value of z in the ε expansion (which is the only additional scaling exponent introduced by the dynamic formalism). Since WF is very standard we will however be somewhat laconic.

The MSR Lagrangian is

$$\mathcal{L} = \psi\left(\partial_t\phi - \nu\nabla^2\phi + r\phi + \frac{g}{3!}\phi^3\right) - D\psi^2 = \frac{1}{2}\Gamma^T G_0^{-1}\Gamma - \frac{g}{3!}\psi\phi^3, \tag{6}$$

[1] Here we are writing things for a non-conserved order parameter; in different situations one would replace the diffusion constant D by an appropriate differential operator.

[2] Warning: I have not made a proper effort of keeping track of minus signs in this section.

where $\Gamma = (\phi,\, \psi)^T$ and the free propagator G_0 is, in momentum space,

$$G_0(q,\omega) = \begin{pmatrix} \frac{2D}{\omega^2+(r+\nu q^2)^2} & \frac{1}{i\omega+r+\nu q^2} \\ \frac{1}{-i\omega+r+\nu q^2} & 0 \end{pmatrix}. \tag{7}$$

We now integrate out the fast modes at one-loop order. Since renormalization of ν only sets in at 2-loop order (the sunrise diagram), we will set $\nu = 1$. The same reasoning applies to D, and we will likewise set $D = 1$. The term $\langle \mathcal{L}_{<>} \rangle_>$ only renormalizes r; the relevant diagram has a $\langle \phi\phi \rangle$ loop attached to a $\langle \phi\psi \rangle$ leg. Letting $\Gamma_>$ have support on all frequencies and on a momentum shell $\Lambda - d\Lambda \leq q \leq \Lambda$, this gives, to quadratic order in g, r,

$$\mathcal{L}_{\rm eff} \supset 2\psi_< \phi_< g \int_{q\in\text{shell}} \int_\omega \frac{1}{\omega^2 + (r+q^2)^2} \to \psi_< \phi_< g A_d \left(\Lambda^{d-2} - r\Lambda^{d-4}\right) d\ln\Lambda, \tag{8}$$

where A_d is the surface area of the unit $d-1$ sphere divided by $(2\pi)^d$.

g is renormalized at 1-loop by a bubble diagram built from a $\langle \phi\phi \rangle$ line and a $\langle \phi\psi \rangle$ line. Setting the external momentum and frequencies to zero since corrections to this are irrelevant, this gives (dropping $O(rg^2)$ terms)

$$\mathcal{L}_{\rm eff} \supset -\psi_< \phi_<^3 C\lambda^2 \int_{q\in\text{shell}} \int_\omega \frac{1}{(-i\omega + q^2)(\omega^2+q^4)} = -\psi_< \phi_<^3 \frac{Cg^2 A_d}{4} \Lambda^{d-4} d\ln\Lambda \tag{9}$$

where I think the combinatorial factor is $C = 6$ (in fact I may be off by a factor of 2; for this particular example I will not bother to investigate).

Taking $d = 4 - \varepsilon$ and working to linear order in ε, the effective couplings after integrating out the fast modes are thus (using $A_4 = 1/8\pi^2$)

$$\begin{aligned} r_{\rm eff} &= r\left(1 + \frac{g}{8\pi^2}(1-r)dl\right) \\ g_{\rm eff} &= g\left(1 - \frac{3g}{8\pi^2}dl\right) \end{aligned} \tag{10}$$

where $dl \equiv d\ln\Lambda$ is the RG time step.

We now rescale momenta to return the cutoff to its original value, by sending $\psi(k,\omega) \to (1 + d_\psi dl)\psi(k',\omega')$ and $\phi(k,\omega) \to (1 + d_\phi dl)\phi(k',\omega')$, where $k' = (1+dl)k, \omega' = (1+zdl)\omega$ and z, d_ϕ, d_ψ are to be determined (this scaling done so that k' is cutoff at Λ to first order in dl). Then

$$\begin{aligned} \frac{dr}{dl} &= r(d + z - d_\phi - d_\psi) + \frac{g}{8\pi^2}(1-r) \\ \frac{dg}{dl} &= g(d + z - 3d_\phi - d_\psi) - \frac{3g^2}{8\pi^2} \\ \frac{dh}{dl} &= h(d + z - d_\psi) \\ \frac{dD}{dl} &= D(d + z - 2d_\psi) \\ \frac{d\nu}{dl} &= \nu(d + z - 2 - d_\psi - d_\phi). \end{aligned} \tag{11}$$

Here we must fix d_ψ by requiring that $\int d^dx\, dt\, \psi\partial_t\phi$ be invariant; this fixes $d_\psi = d - d_\phi$. For ν to be invariant, we must choose $z = 2$, and so we conclude that

$$z = 2 + O(\varepsilon^2). \tag{12}$$

At higher orders it turns out that $z = 2 + x$ with x positive and rather small; the best estimates from the literature appear to give $x \approx 0.14$ when $\varepsilon = 2$.[3] Continuing, we take D to be fixed by letting

[3] We know that x can be negative in non-equilibrium settings (e.g. for KPZ in 1d, where $x = -1/2$). Must it be positive in equilibrium?

$d_\psi = d/2+1$, this makes $d_\phi = d/2-1 = 1-\varepsilon/2$. This then fixes everything and yields the usual WF beta functions. The only point of this exercise is thus to show that z is not corrected away from the typical diffusive result at one-loop order.

warmup II: KPZ

We now look at KPZ. The beta functions were of course given in the original reference, but I have not seen an explicit derivation of them, and hence will provide one below. We will follow the notation of the original paper and write the Langevin equation as

$$\partial_t \phi = \nu \nabla^2 \phi + \frac{\lambda}{2}(\nabla \phi)^2 + \eta. \tag{13}$$

We note in passing that the paper [1] claims that *all* equilibrium-breaking perturbations to dynamics with a non-conserved Ising-like order parameter are irrelevant (regardless of symmetry considerations). This paper however only considers perturbations that are marginal at $d=4$, viz. $\psi\phi^2$ (generically giving a first-order transition) and $\psi \mathbf{E}\cdot\phi\nabla\phi$ for some fixed vector field $\mathbf{E}$.[4] By contrast, the KPZ-type terms considered here are instead irrelevant in $d=4$ (more generally, they are irrelevant whenever ϕ has positive scaling dimension, which is of course always true at the upper critical dimension). This fact means that performing an ε expansion to one loop is slightly unsavory, since any non-irrelevance can only be achieved by a term of $O(\varepsilon)$ growing larger than a term of $O(\varepsilon^0)$ (making the neglect of $O(\varepsilon^2)$ terms especially egregious), but we will ignore this and proceed with the calculation regardless (this is after all what was done in the original KPZ paper).

The Lagrangian in the MSR formalism is[5]

$$\mathcal{L} = \psi\left(\partial_t\phi - \nu\nabla^2\phi - \frac{\lambda}{2}(\nabla\phi)^2\right) - D\psi^2 = \frac{1}{2}\Gamma^T G_0^{-1}\Gamma - \frac{\lambda}{2}\psi(\nabla\phi)^2, \tag{14}$$

where G_0 is now

$$G_0(q,\omega) = \begin{pmatrix} \frac{2D}{\omega^2+\nu^2q^4} & \frac{1}{i\omega+\nu q^2} \\ \frac{1}{-i\omega+\nu q^2} & 0 \end{pmatrix}. \tag{15}$$

We now split ψ and ϕ into fast and slow modes. The linear part $\langle \mathcal{L}_{<>}\rangle_>$ does not do anything because the interaction is cubic. Focusing on the quadratic part, we first find the term which renormalizes D. This diagram has a single bubble made up of two $\langle\phi\phi\rangle$ lines. Neglecting the (irrelevant) dependence on the external momentum by setting it to zero, this gives

$$\begin{aligned} \mathcal{L}_{\text{eff}} &\supset -\psi_<^2\frac{\lambda^2}{4}\int_{q\in\text{shell}}\int_\omega q^4[G_0(q,\omega)]^2_{\phi\phi} = -\psi_<^2\lambda^2D^2\int_{q\in\text{shell}}\int_\omega \frac{q^4}{(\omega^2+\nu^2q^4)^2} \\ &= -\psi_<^2\frac{\lambda^2D^2}{4\nu^3}\int_{q\in\text{shell}} q^{-2} = -\psi_<^2\frac{\lambda^2D^2}{4\nu^3}A_d\Lambda^{d-2}dl. \end{aligned} \tag{16}$$

Next we examine the term that renormalizes ν. This term has two outgoing ϕ legs and a bubble made up of one $\langle\phi\psi\rangle$ propagator and one $\langle\phi\phi\rangle$ propagator. Letting $\mathbf{k}$ denote the momentum of the incoming and outgoing fields (their frequency can be set to zero up to irrelevant terms, but of course thier momentum dependence must be retained for determining the flow of ν), and sending momentum $\mathbf{q}-\mathbf{k}$ through the $\langle\phi\phi\rangle$ propagator, this gives[6]

$$\begin{aligned} \mathcal{L}_{\text{eff}} &\supset \psi_<\phi_< 2\times\frac{\lambda^2}{4}\int_{q\in\text{shell}}\int_\omega \mathbf{k}\cdot(\mathbf{q}-\mathbf{k})\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})[G_0(\mathbf{k}+\mathbf{q},\omega)]_{\phi\psi}[G_0(\mathbf{q},\omega)]_{\psi\phi} \\ &= \psi_<\phi_<\lambda^2D\int_{q\in\text{shell}}\int_\omega\frac{\mathbf{k}\cdot(\mathbf{q}-\mathbf{k})\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})}{(-i\omega+\nu q^2)(\omega^2+\nu^2(\mathbf{q}-\mathbf{k})^4)} \end{aligned} \tag{17}$$

[4] A term like $\mathbf{E}\cdot\nabla\phi$ can always be eliminated by a coordinate change.

[5] We will not need to introduce a coupling for the $\psi\partial_t\phi$ term, since the self energy will turn out to not have any frequency dependence at one-loop order (because the self energy is generated by λ vertices, which vanish when the momentum of ϕ is set to zero).

[6] Here we will be expanding in small k and hence will pretend that $\mathbf{k}+\mathbf{q}$ is always in the shell if $\mathbf{q}$ is.

where the 2 in front comes from the two distinct diagrams. We now do the integral over ω and select out the part proportional to k^2. The denominator after integrating over ω is $\propto (\mathbf{q}-\mathbf{k})^4 + (\mathbf{q}-\mathbf{k})^2 q^2$; we only need this to order k, which gives just $2q^4(1-3\mathbf{q}\cdot\mathbf{k}/q^2)$. Then we have

$$\begin{aligned}\mathcal{L}_{\text{eff}} &\supset -\psi_< \phi_< \frac{\lambda^2 D}{4\nu^2}\int_{q\in\text{shell}} q^{-4}\frac{(\mathbf{q}\cdot\mathbf{k})^2 - q^2\mathbf{k}\cdot\mathbf{q} + k^2 q^2}{1-3\mathbf{q}\cdot\mathbf{k}/q^2} \\ &\to -\psi_<\phi_< k^2\frac{\lambda^2 D}{4\nu^2}\int_{q\in\text{shell}}\frac{k^2q^2 - 2(\mathbf{k}\cdot\mathbf{q})^2}{q^4} \\ &= -\psi_<\phi_< k^2 \frac{\lambda^2 D}{4\nu^2}\frac{d-2}{d} A_d \Lambda^{d-2} dl.\end{aligned} \tag{18}$$

We are actually done, since we do not need to consider the renormalization of λ. KPZ just say that the diagrams renormalizing λ cancel; this is not a coincidence and is in fact due to the boost-ish symmetry[7]

$$\phi(\mathbf{x},t) \mapsto \phi(\mathbf{x}+\lambda t\mathbf{u}, t) + \mathbf{u}\cdot\mathbf{x} + \frac{\lambda}{2}u^2 t. \tag{19}$$

Since this is a symmetry of the UV stochastic equation, with the value λ fixed at its UV value, the renormalization of λ is trivial.

Taking this into account, after integrating out the fast modes, the effective couplings are

$$\nu_{\text{eff}} = \nu\left(1 + \frac{\lambda^2 D}{4\nu^3}\frac{2-d}{d}A_d\Lambda^{d-2}dl\right), \qquad D_{\text{eff}} = D\left(1+\frac{\lambda^2 D}{4\nu^3}A_d\Lambda^{d-2}dl\right). \tag{20}$$

After rescaling and using the above expressions for $\nu_{\text{eff}}, D_{\text{eff}}$, we obtain the RG equations (choosing units so that $\Lambda = 1$)

$$\begin{aligned}\frac{d\nu}{dl} &= \left(z - 2 + A_d\frac{\lambda^2 D}{4\nu^3}\frac{2-d}{d}\right)\nu \\ \frac{dD}{dl} &= \left(z - d + 2d_\phi + A_d\frac{\lambda^2 D}{4\nu^3}\right) D \\ \frac{d\lambda}{dl} &= (z-2-d_\phi)\lambda\end{aligned} \tag{21}$$

which exactly match those of KPZ (their χ is our $-d_\phi$). Following KPZ, define the effective coupling

$$\bar\lambda \equiv \lambda\sqrt{\frac{D}{\nu^3}}. \tag{22}$$

The scaling here is reasonable: the interactions should get stronger when the noise strength (D) is larger, and should get weaker when the diffusion constant (ν) is larger, since stronger diffusion means correlations decay more rapidly (and thus suppress the typical magnitude of ϕ). The exact dependence on D and ν can be obtained by performing a rescaling on the KPZ equation to set $D = \nu = 1$ (in $\Lambda = 1$ units); doing so reveals that the nonlinearity is controlled by $\bar\lambda$.

The beta function of $\bar\lambda$ is

$$\frac{d\bar\lambda}{dl} = (1-d/2)\bar\lambda + A_d\left(\frac{1}{2} - \frac{3}{4d}\right)\bar\lambda^3. \tag{23}$$

Thus $\bar\lambda$ is irrelevant in $d > 2$, marginally relevant in $d = 2$, and relevant in $d = 1$.

In $d = 2$ we cannot track the flow from the 1-loop equations: ν being stationary mandates $z = 2$, but then the flow of D runs away. In $d = 1$ we get $A_1\bar\lambda^2/4 = 2 - z = 1 - z - 2d_\phi$, which gives $d_\phi = -1/2$ and the famous $z = 3/2$ (where for the reasons discussed by KPZ this result is exact).

[7] This is a "boost-ish" symmetry because it generates a Galilean boost for the "velocity" field $v_j = \nabla_j\phi$.

Anisotropic KPZ

Now we consider a variant of KPZ in $d > 1$ for which the nonlinearity is anisotropic:

$$\sum_{a=1}^{n}(\partial_a\phi)^2 - \sum_{b=n+1}^{d}(\partial_b\phi)^2. \tag{24}$$

It is known that this nonlinearity is actually irrelevant for $d = 2, n = 1$ as stated in Wolf 99; here we will provide the details of the derivation.

Consider first the renormalization of D. The relevant integral is, letting $q_\perp^2 = \sum_{a=1}^n q_a^2$ and $q_\parallel^2 = \sum_{b=1}^n q_b^2$,

$$2\lambda^2 D^2 \int_{q\in\text{shell}} \int_\omega \frac{(q_\perp^2 - q_\parallel^2)^2}{(\omega^2 + \nu^2 q^4)^2} = \frac{\lambda^2 D^2}{2\nu^3} B_{d,n} \Lambda^{d-2} d\ln\Lambda \tag{25}$$

where

$$B_{d,n} \equiv \int_{S^{d-1}} \frac{d\Omega}{(2\pi)^d}(1 - 2\langle\Omega|\Pi_\perp|\Omega\rangle)^2, \tag{26}$$

with $\Pi_\perp$ the projector onto the first n coordinates. For example,

$$B_{2,1} = \frac{1}{4\pi^2}\int d\theta\,(1 - 2\cos^2\theta)^2 = \frac{1}{4\pi}. \tag{27}$$

Now for the renormalization of ν. The frequency integration and expansion of the denominator is done in the same way as before, giving (remembering the 2 from the symmetry of the diagram)

$$\begin{aligned}
&\frac{\lambda^2 D}{\nu^2}\int_{q\in\text{shell}} q^{-4}(1 + 3\mathbf{q}\cdot\mathbf{k}/q^2)\left((-k_\parallel^2 + k_\perp^2 + \mathbf{k}_\parallel\cdot\mathbf{q}_\parallel - \mathbf{k}_\perp\cdot\mathbf{q}_\perp)(q_\parallel^2 - q_\perp^2) - (\mathbf{k}_\parallel\cdot\mathbf{q}_\parallel - \mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2\right) \\
&\quad\to \frac{\lambda^2 D}{\nu^2}\int_{q\in\text{shell}} q^{-4}\Bigg((q_\parallel^2 - q_\perp^2)(-k_\parallel^2 + k_\perp^2 + 3q^{-2}((\mathbf{k}_\parallel\cdot\mathbf{q}_\parallel)^2 - (\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2)) \\
&\qquad - (\mathbf{k}_\parallel\cdot\mathbf{q}_\parallel)^2 - (\mathbf{k}_\perp\cdot\mathbf{q}_\perp)^2\Bigg)
\end{aligned} \tag{28}$$

From this we see that if $n \neq d/2$, we will have to introduce $\nu_\perp$ and $\nu_\parallel$, and compute their RG equations separately (since then $\int_{q\in\text{shell}}(q_\parallel^2 - q_\perp^2) \neq 0$). Since this sounds like more effort that it is worth, we will henceforth assume $d = 2n$. Then the above integral becomes, after some algebra,

$$-\frac{2\lambda^2 D}{d\nu^2} k^2 G_d \Lambda^{d-2} d\ln\Lambda \tag{29}$$

where

$$G_d \equiv -\int_{S^{d-1}} \frac{d\Omega}{(2\pi)^d}(2\langle\Omega|\Pi_\perp|\Omega\rangle\langle\Omega|\Pi_\parallel|\Omega\rangle - \langle\Omega|\Pi_\perp|\Omega\rangle^2). \tag{30}$$

In 2d,

$$G_2 = -\frac{1}{4\pi^2}\int d\theta\,(2\sin^2\theta\cos^2\theta - \cos^4\theta) = \frac{1}{16\pi}. \tag{31}$$

Finally, we have the renormalization of λ. I believe that this however is trivial for a similar symmetry-based reason as in conventional KPZ. Here the transformation one considers is now

$$\phi(\mathbf{x}, t) \mapsto \phi(\mathbf{x} + \lambda t\widetilde{\mathbf{u}}, t) + \frac{\lambda}{2}t\left(\sum_{a=1}^{n} u_a^2 - \sum_{b=n+1}^{d} u_b^2\right) + \mathbf{u}\cdot\mathbf{x}, \tag{32}$$

where $\widetilde{u}_a \equiv s_a u_a$, with $s_a = +1$ ($s_a = -1$) if $a \leq n$ ($a > n$). Since this is a symmetry of the UV Langevin equation with the UV value of λ, the latter can only renormalize due to spacetime rescalings.

The RG equations are then

$$\begin{aligned}
\frac{d\ln\nu}{dl} &= z-2+\frac{2\bar{\lambda}^2}{d}G_d \\
\frac{d\ln D}{dl} &= z-d+2d_\phi+\frac{\bar{\lambda}^2}{4}B_{d,d/2} \\
\frac{d\ln\bar{\lambda}}{dl} &= 1-d/2+\bar{\lambda}^2\left(-\frac{3}{d}G_d+\frac{B_{d,d/2}}{8}\right),
\end{aligned} \tag{33}$$

where $\bar{\lambda}^2=\lambda^2D/\nu^3$ as before.

Consider $d=2, n=1$. Then we get

$$\begin{aligned}
\frac{d\ln\nu}{dl} &= z-2+\frac{\bar{\lambda}^2}{16\pi} \\
\frac{d\ln D}{dl} &= z-d+2d_\phi+\frac{\bar{\lambda}^2}{16\pi} \\
\frac{d\ln\bar{\lambda}}{dl} &= -\frac{\bar{\lambda}^2}{16\pi}
\end{aligned} \tag{34}$$

where amazingly the first equation exactly matches the one quoted in the aforementioned paper by Wolf (although given how sloppy I have been with factors of 2 this is likely an accident; also the RHS of the third equation differs by a factor of 2 [although the sign is correct, which is the important part]). Note that we now have a nontrivial contribution to $d\ln\nu/dl$ even in 2d, which was not present in the isotropic case. More importantly, note that $\bar{\lambda}$ is now marginally *irrelevant*.[8] Therefore we expect to find a simple $z=2, d_\phi=0$ free Gaussian fixed point. Of course this conclusion needs to be accompanied by the (large) caveat that we are only working to $O(\varepsilon)$ and are setting $\varepsilon=O(1)$; I thus see no particular reason why this conclusion should survive to higher loop orders.

References

[1] K. Bassler and B. Schmittmann. Critical dynamics of nonconserved ising-like systems. *Physical review letters*, 73(25):3343, 1994.

[8] Changing the sign of the anisotropy can only modify the $O(\bar{\lambda}^2)$ and higher terms in the β functions, since the scaling dimension of the operator $(\partial_x\phi)^2+\zeta(\partial_y\phi)^2$ is independent of ζ.