# Comments on metastability in the active Ising model

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It was recently appreciated in (Benvegnen,. . . ,Tailleur 23) that the flocking phase of the active Ising model of Solon and Tailleur is in fact metastable: a large enough (and dense enough) droplet of minority spins, once nucleated, grows uncontrollably by forming a shock wave feature. In this short note we make two remarks on this phenomenon.

### *Mechanism and avoidance with repulsive interactions*

Let us try to understand the development of the instability at a heuristic level. Consider a front where a density $\rho_+$ of + spins collide with a density $\rho_- < \rho_+$ of − spins. Let $\gamma$ be the rate at which the spins on each site thermalize according to their mean-field-like Ising dynamics. If $\gamma$ is much greater than the rate at which spins from the + flock move into the − flock, we do not expect the + flock to grow, since + spins that infiltrate the − flock will thermalize to become − before they can "overwhelm" the other − spins. On the other hand, if $\gamma$ is much less than this rate, the + spins will convert the − spins to +, and the flock will grow. This suggests that as long as $\rho_+ - \rho_-$ is sufficiently large (how large depends on $\gamma$), the + flock will grow. If this intuition is correct, it predicts that metastability will be eliminated (in a certain region of parameter space) if an upper cutoff is placed on the number of particles per site, or if the particles are replaced by e.g. repulsively-interacting discs in an off-lattice setting.

### *Independence on relative flying directions*

Consider more generically a situation where + spins prefer to move along $\mathbf{v}^+$ and − spins prefer to move along $\mathbf{v}^-$, with the usual AIM taking $\mathbf{v}^+ = -\mathbf{v}^-$. One might wonder if taking e.g. $\mathbf{v}^+ \perp \mathbf{v}^-$ could favor stability, as two opposite flocks will in then not meet "head-on", with instead one flock "ambushing" another from the side. This however turns out not to be the case, which is fairly reasonable given the discussion above.

To show this numerically in the hydrodynamic regime, we numerically integrate the hydro equations for the number densities of + and − spins $n^\pm$. The hydro equations are

$$\partial_t n^\pm = D\nabla^2 n^\pm + \mathbf{v}^\pm \cdot \nabla n^\pm \pm n^- e^{\beta m/\rho} \mp n^+ e^{-\beta m/\rho} \tag{1}$$

where the last terms try to align the spins on each site. Taking the symmetric and antisymmetric combinations of these equations,

$$\begin{aligned} \partial_t \rho &= D\nabla^2 \rho + \mathbf{u} \cdot \nabla \rho + \mathbf{w} \cdot \nabla m \\ \partial_t m &= D\nabla^2 m + \mathbf{u} \cdot \nabla m + \mathbf{w} \cdot \nabla \rho + 2(\rho \sinh(\beta m/\rho) - m \cosh(\beta m/\rho)) \end{aligned} \tag{2}$$

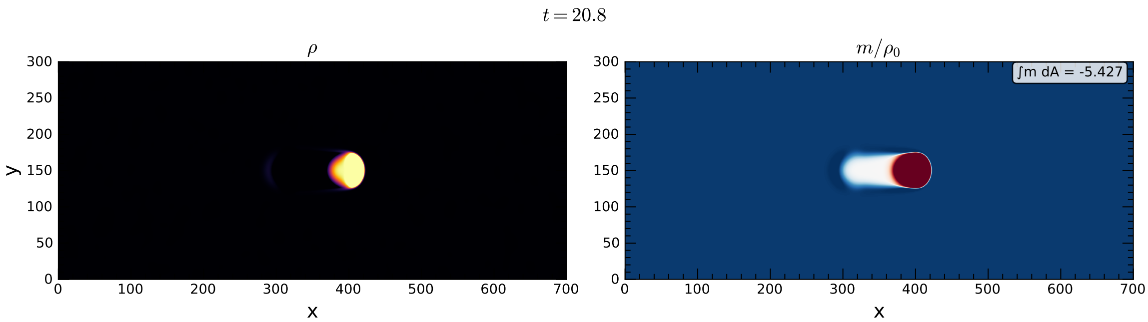
where

$$\mathbf{u} = \mathbf{v}^+ + \mathbf{v}^-, \qquad \mathbf{w} = \mathbf{v}^+ - \mathbf{v}^-. \tag{3}$$

The new feature of these equations is the addition of the terms proportional to $\mathbf{u}$, but this turns out to be innocuous.

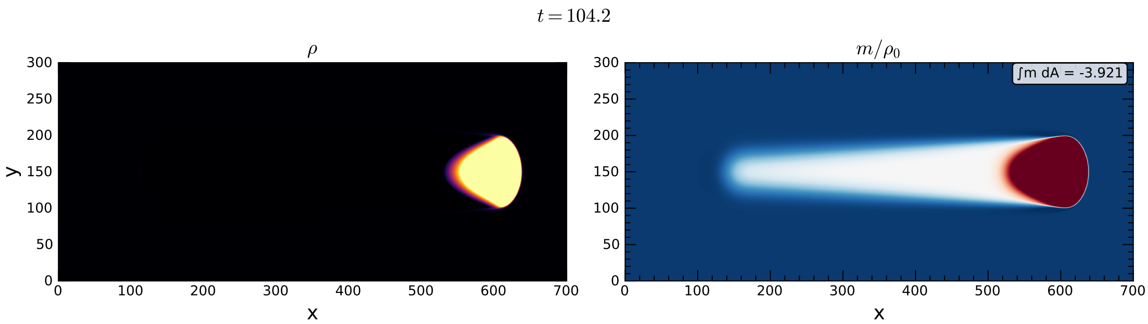
We investigate this by simulating these equations starting from a configuration with a very dense minority droplet in a homogeneous background and tracking how big the droplet gets. We will set $\beta = 2, D = 1, v = 1$, take a background density of $\rho_0 = 6$, a bubble radius of 20, a droplet density of $3\rho_0$, and set $\mathbf{v}^- = v\hat{\mathbf{x}}$.

For $\mathbf{v}^+ = -v\hat{\mathbf{x}}$, we find the same droplet profile as in the aforementioned paper: at early times,



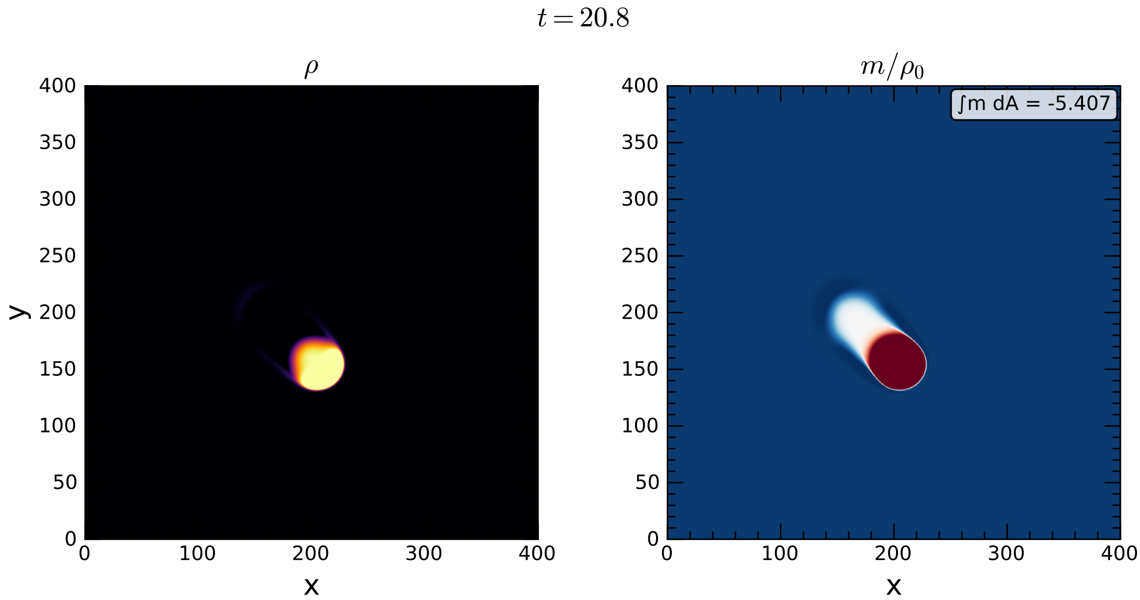

(4)

while at later times



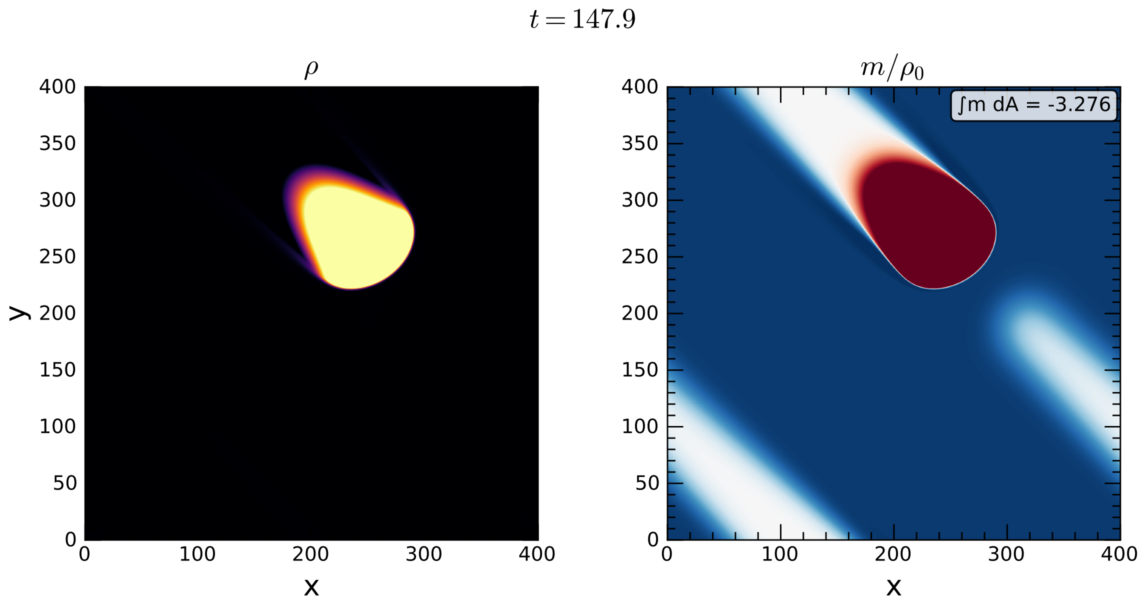

(5)

Now take $\mathbf{v}^+ = -v\hat{\mathbf{y}}$. The droplet still develops a tail and shock front; the only change is that the center of the droplet moves ballistically along $-\hat{\mathbf{y}}$. At early times:




(6)

while at later times:

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